Degeneracy of the action-frequency map: A mechanism for homoclinic bifurcation of invariant tori

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A family of smooth invariant tori of a Hamiltonian system can be parameterized by the values of the actions or the frequencies. These parameterizations are related by the action-frequency map. The purpose of this paper is to show that when the action-frequency map is degenerate, it signals a homoclinic bifurcation. Remarkably, the nonlinear properties of this homoclinic bifurcation to invariant tori are determined by the curvature of the action-frequency map. A homoclinic angle is also generated which is analogous to a Hannay-Berry phase shift. The theory is constructive and so can usefully be combined with computation. Some implications for quantization, and the generation of solitary waves are also discussed.

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I. INTRODUCTION

Invariant tori are a ubiquitous and important class of solutions of Hamiltonian dynamical systems [1,2]. For example, stable equilibrium points and stable periodic orbits are surrounded by invariant tori, and tori are typical solutions of integrable systems. Invariant tori are also fundamental in quantization [1,3]. In this paper a mechanism for the generation of a homoclinic orbit, which is bi-asymptotic to the torus, is presented. This bifurcation is of interest for several reasons. It shows how the geometry of the action-frequency map, which is a central part of any constructive theory, encodes information about homoclinic bifurcation. It has implications for semiclassical quantization. Both the quantization of tori and the quantization of homoclinic orbits have been extensively studied (cf. [3–7] and references therein), and the quantum implications of the saddle-center bifurcation of periodic orbits has been studied [6]. The theory of this paper provides a mechanism for the transition between tori, through a saddle-center bifurcation, to a homoclinic orbit. The theory is also of interest in pattern formation. It gives a mechanism for the creation of toral dark solitary waves (an example is given in Sec. VII).

In the classical setting, orbits homoclinic to invariant tori have been studied (e.g., [8–10]), but these studies do not consider the mechanism for creating the homoclinic orbit. An early work on the bifurcation of homoclinic orbits through saddle-center bifurcation of invariant tori is Hanßmann [11]; for a recent survey and follow-up references, see the recent book [12]. Take a $(2n+2)$-dimensional phase space with coordinates $(\theta_1,\ldots,\theta_n,I_1,\ldots,I_n,q,p)$. Hanßmann takes the standard Hamiltonian function for the saddle-center bifurcation

$$ap^2+\frac{b}{3}q^3-\lambda q,$$

with parameters a,b,λ and attaches a *nondegenerate* torus

$$H(q,p,I)=\boldsymbol{\omega}\cdot I+a(\boldsymbol{\omega})p^2+\frac{b(\boldsymbol{\omega})}{3}q^3-\lambda q, \tag{1}$$

allowing the parameters a,b to depend on the frequency vector. He then proceeds to study the effect of perturbations on the persistence of the invariant tori. A key hypothesis in this work (and all follow-up work) is that the tori are nondegenerate.

In this paper a mechanism is identified based on *degeneracy* of the invariant tori. No external parameters are required.

Assume throughout the paper that the dimension of the phase space is $2n+2$ as this is the lowest dimension that the phenomena occurs for a torus of dimension n.

II. GEOMETRY OF THE ACTION-FREQUENCY MAP

With the appropriate smoothness, n-dimensional tori naturally arise in an n-parameter family. It will be assumed throughout that the Hamiltonian function is sufficiently smooth and the system is integrable (or if not integrable, normal form transformations have been carried out to sufficient order such that the leading-order truncated system is integrable; nonintegrability and persistence are briefly discussed in Sec. VI).

In the neighborhood of nondegenerate points, the tori can be parameterized by the frequencies $(\omega_1,\ldots,\omega_n)$ or the actions $(I_1,\ldots,I_n)$. These two parameterizations are related by the action-frequency map $I(\boldsymbol{\omega}):=[I_1(\boldsymbol{\omega}),\ldots,I_n(\boldsymbol{\omega})]$ (cf. Sec. 6.1 of [1]). The torus is said to be nondegenerate when

$$\det[\mathrm{D}I(\boldsymbol{\omega})]\neq 0, \tag{2}$$

where $\mathrm{D}I(\boldsymbol{\omega})$ is the Jacobian

$$\mathrm{D}I(\boldsymbol{\omega}):=\begin{pmatrix}\frac{\partial I_1}{\partial\omega_1} & \cdots & \frac{\partial I_1}{\partial\omega_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial I_n}{\partial\omega_1} & \cdots & \frac{\partial I_n}{\partial\omega_n}\end{pmatrix}. \tag{3}$$

Note that the Kolmogorov-Arnold-Moser (KAM) nondegeneracy condition is normally stated for the *inverse map*: $\det[\mathrm{D}\boldsymbol{\omega}(I)]\neq 0$ and the two conditions are equivalent at nondegenerate points.

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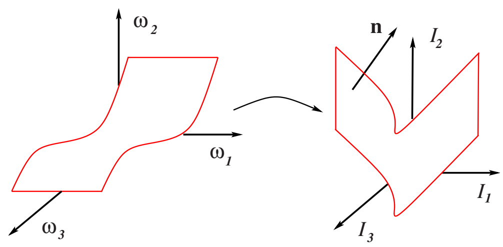

FIG. 1. (Color online) Example with $n=3$ showing a singular surface in frequency space and its image under $I(\omega)$ in action space.

The action-frequency map is *degenerate* when

$$\det[\mathrm{D}I(\boldsymbol{\omega})]=0. \tag{4}$$

Assume simple degeneracy throughout ($\mathrm{rank}=n-1$). Then Eq. (4) defines a surface of dimension $n-1$ in frequency space—the *frequency hypersurface*. An example with $n=3$ is shown in Fig. 1. There will also be a hypersurface in action space defined by the image of $I(\boldsymbol{\omega})$, and it will be called the *action hypersurface*.

The main result of this paper is to show that there is a homoclinic bifurcation—in phase space—for values of the action near the action hypersurface. To show this, the geometry of the action and frequency hypersurfaces need to be studied in more detail.

Figure 1 shows a normal vector $\mathbf{n}$ to the action hypersurface. It is the eigenvector associated with the zero eigenvalue of the Jacobian

$$\mathrm{D}I(\boldsymbol{\omega})\mathbf{n}=0. \tag{5}$$

To see the geometric interpretation of $\mathbf{n}$, note that tangent vectors in frequency space, denoted by $\dot{\boldsymbol{\omega}}$, are related to tangent vectors in action space, $\dot{I}$, by

$$\dot{I}=\mathrm{D}I(\boldsymbol{\omega})\dot{\boldsymbol{\omega}}.$$

Hence, since $\mathrm{D}I(\boldsymbol{\omega})$ is symmetric, it follows that $\mathbf{n}$ is perpendicular to $\dot{I}$, and so is normal to the action hypersurface.

A surprising result is that the second derivative of the action-frequency map, in the following form, appears in an important way in determining the properties of the homoclinic bifurcation. Not any second derivative, but the second derivative in the direction $\mathbf{n}$—that is, transverse to the action hypersurface: define

$$\hat{\kappa}=\left.\frac{d^2}{ds^2}\right|_{s=0}\mathcal{A}(\boldsymbol{\omega}+s\mathbf{n}), \tag{6}$$

where

$$\mathcal{A}(\boldsymbol{\omega}):=\mathbf{n}\cdot[I(\boldsymbol{\omega})-I(\boldsymbol{\omega}_0)], \tag{7}$$

where $\boldsymbol{\omega}_0$ is any point on the singular hypersurface. This latter definition is taken with $\mathbf{n}$ fixed at the point ω_0 on the singular hypersurface. $\mathcal{A}(\boldsymbol{\omega})$ is a measure of the distance away from the singular hypersurface in action space.

So far, just the geometry of the mapping $I(\boldsymbol{\omega})$ has been studied. In order to relate this geometry to the homoclinic bifurcation, it needs to be connected to the dynamics. A key ingredient in this connection is Percival's variational principle for invariant tori [13].

III. HAMILTONIAN SYSTEMS AND INVARIANT TORI

Consider a nonlinear autonomous Hamiltonian system

$$\mathbf{q}_t=\frac{\partial H}{\partial \mathbf{p}},\quad \mathbf{p}_t=-\frac{\partial H}{\partial \mathbf{q}},\quad \begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix}\in\mathbb{R}^{2n+2}, \tag{8}$$

where $H(\mathbf{q},\mathbf{p})$ is a given Hamiltonian function. Attention is restricted to $(2n+2)$-dimensional phase space as it is the lowest dimension in which the bifurcation occurs. Moreover, to avoid technicalities with small divisors, assume that the system is integrable.

Consider a n-parameter family of invariant tori,

$$[\mathbf{q}(t),\mathbf{p}(t)]:=[\hat{\mathbf{q}}(\theta,\omega),\hat{\mathbf{p}}(\theta,\omega)], \tag{9}$$

with frequencies $\omega_1,\ldots,\omega_n$,

$$\theta:=(\theta_1,\cdots,\theta_n),$$

$$\theta_j=\omega_j t+\theta_j^0,$$

where θ_j^0 is an arbitrary phase shift, and $(\hat{\mathbf{q}},\hat{\mathbf{p}})$ are 2π-periodic functions in each θ_j. Substituting Eqs. (9) and (8) leads to

$$-\sum_{j=1}^{n}\omega_j\frac{\partial\hat{\mathbf{p}}}{\partial\theta_j}=\frac{\partial H}{\partial\mathbf{q}},\quad \sum_{j=1}^{n}\omega_j\frac{\partial\hat{\mathbf{q}}}{\partial\theta_j}=\frac{\partial H}{\partial\mathbf{p}}. \tag{10}$$

These equations can be interpreted as the Euler-Lagrange equation for Percival's variational principle [13]. Define actions

$$I_j(\omega):=\oint\hat{\mathbf{p}}\cdot\frac{\partial\hat{\mathbf{q}}}{\partial\theta_j}d\theta, \tag{11}$$

where

$$\oint f(\theta)d\theta:=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}f(\theta)d\theta_1\cdots d\theta_n, \tag{12}$$

and define $\mathcal{H}$ to be the average of $H(\hat{\mathbf{q}},\hat{\mathbf{p}})$ over the torus. Then Eq. (10) is the Euler-Lagrange equation for

$$\mathcal{L}(\hat{\mathbf{q}},\hat{\mathbf{p}},\boldsymbol{\omega})=\mathcal{H}-\sum_{j=1}^{n}\omega_j I_j,$$

with $\omega_1,\ldots,\omega_n$ treated as Lagrange multipliers. A standard result in the theory of Lagrange multipliers is that the constrained variational principle is nondegenerate precisely when condition (2) is satisfied with I_j interpreted as values of the constraint sets and $\omega_1,\ldots,\omega_n$ interpreted as Lagrange multipliers. This connects Percival's variational principle with geometric condition (2).

Define the second variation in Percival's variational principle by

$$\mathbf{L}(\theta,\omega) := \mathrm{D}^2\mathcal{H} - \sum_{j=1}^{n} \omega_j \mathrm{D}^2 I_j .$$

Then the linearization of Eq. (8) about the family of invariant tori can be written in the following form:

$$\mathbf{J}\mathbf{u}_t = \mathbf{L}(\theta,\omega)\mathbf{u}, \quad \mathbf{u} = (\mathbf{q},\mathbf{p}), \tag{13}$$

where $\mathbf{J}$ is a unit symplectic operator.

The next step is to show that the linearization about the invariant torus has a zero eigenvalue of geometric multiplicity n, and algebraic multiplicity $2n$, and the algebraic multiplicity jumps to $2n+2$ if and only if condition (4) is satisfied. The special case $n=1$ (periodic orbits) has been proved in [14] and the generalization to arbitrary n follows similar lines. The related problem of degenerate relative equilibria is given in [15]. A sketch of the argument is given here.

Let $\hat{\mathbf{u}} := (\hat{\mathbf{q}},\hat{\mathbf{p}})$ and define

$$\mathbf{v}_j = \frac{\partial \hat{\mathbf{u}}}{\partial \theta_j}.$$

Differentiate the Euler-Lagrange [Eq. (10)] with respect to θ_j for $j=1,\ldots,n$; then

$$\mathbf{L}(\theta,\omega)\mathbf{v}_j = 0, \quad j=1,\ldots,n.$$

This construction confirms that zero is an eigenvalue of $\mathbf{L}(\theta,\omega)$ of algebraic and geometric multiplicity of at least n. Now differentiate Eq. (10) with respect to ω_ℓ, $\ell=1,\ldots,n$,

$$-\sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \theta_j}\frac{\partial \hat{\mathbf{p}}}{\partial \omega_\ell} = H_{\mathbf{qq}}\frac{\partial \hat{\mathbf{q}}}{\partial \omega_\ell} + H_{\mathbf{qp}}\frac{\partial \hat{\mathbf{p}}}{\partial \omega_\ell} + \frac{\partial \hat{\mathbf{p}}}{\partial \theta_\ell},$$

$$\sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \theta_j}\frac{\partial \hat{\mathbf{q}}}{\partial \omega_\ell} = H_{\mathbf{pq}}\frac{\partial \hat{\mathbf{q}}}{\partial \omega_\ell} + H_{\mathbf{pp}}\frac{\partial \hat{\mathbf{p}}}{\partial \omega_\ell} - \frac{\partial \hat{\mathbf{q}}}{\partial \theta_\ell},$$

or with

$$\mathbf{v}_{n+j} = \frac{\partial \hat{\mathbf{u}}}{\partial \omega_j},$$

this gives

$$\mathbf{L}(\theta,\omega)\mathbf{v}_{n+j} = \mathbf{J}\mathbf{v}_j, \quad j=1,\ldots,n.$$

Hence the $\mathbf{v}_{n+j}$ for $j=1,\ldots,n$ are generalized eigenfunctions of $\mathbf{J}^{-1}\mathbf{L}(\theta,\omega)$. This argument confirms that zero is an eigenvalue of algebraic multiplicity at least $2n$. Using standard Jordan chain theory, the algebraic multiplicity is $2n+2$ if there are two more generalized eigenfunctions. Indeed, it is found that

$$\mathbf{L}(\theta,\omega)\mathbf{v}_{2n+1} = \sum_{j=1}^{n} \mathbf{n}_j \mathbf{J}\mathbf{v}_{n+j},$$

$$\mathbf{L}(\theta,\omega)\mathbf{v}_{2n+2} = \mathbf{J}\mathbf{v}_{2n+1}, \tag{14}$$

where $\mathbf{n}_j$ are the components of the normal vector $\mathbf{n}$. To see that these are the correct terms, first note that Jordan chain theory says that zero is an eigenvalue of algebraic multiplicity $2n+1$ if there exists a generalized eigenfunction $\mathbf{v}_{2n+1}$ satisfying

$$\mathbf{J}^{-1}\mathbf{L}(\theta,\omega)\mathbf{v}_{2n+1} = \sum_{j=1}^{n} a_j \mathbf{v}_{n+j},$$

for some constants $a_1,\ldots,a_n$. Since $\mathbf{L}$ is formally self-adjoint, solvability requires

$$\sum_{j=1}^{n} a_j \langle\langle \mathbf{v}_\ell, \mathbf{J}\mathbf{v}_{n+j}\rangle\rangle = 0, \quad \text{for } \ell=1,\ldots,n,$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ is an inner product including integration over the torus,

$$\langle\langle \mathbf{u},\mathbf{v}\rangle\rangle := \oint \langle \mathbf{u}(\theta),\mathbf{v}(\theta)\rangle \mathrm{d}\theta,$$

with $\langle\cdot,\cdot\rangle$ a standard inner product on $\mathbb{R}^{2n+2}$. But,

$$\langle\langle \mathbf{v}_\ell, \mathbf{J}\mathbf{v}_{n+j}\rangle\rangle = \left\langle\!\!\left\langle \frac{\partial \hat{\mathbf{u}}}{\partial \theta_\ell}, \mathbf{J}\frac{\partial \hat{\mathbf{u}}}{\partial \omega_j}\right\rangle\!\!\right\rangle = -\frac{\partial I_\ell}{\partial \omega_j}.$$

Hence the solvability condition is

$$\sum_{j=1}^{n} a_j \frac{\partial I_\ell}{\partial \omega_j} = 0, \quad \ell=1,\ldots,n,$$

or

$$\mathrm{D}I(\boldsymbol{\omega})\mathbf{a} = 0.$$

Hence if we choose $\mathbf{a}=\mathbf{n}$ then we have established a precise connection between degeneracy (4) and existence of a zero eigenvalue of algebraic multiplicity $2n+2$. The second equation of Eq. (14) follows from the fact that for linear Hamiltonian systems, zero is an eigenvalue of even multiplicity (i.e., algebraic multiplicity $2n+1$ implies $2n+2$).

In the theory of Hanßmann [11] the linear saddle-center decouples from the torus; hence the eigenvalue zero there also has algebraic multiplicity $2n+2$ but the geometric multiplicity is $n+1$, whereas here the geometric multiplicity is just n.

IV. NORMAL FORM THEORY

The idea is to choose new coordinates for the linear system, and then do weakly nonlinear theory to get the nonlinear normal form transverse to the torus. It generalizes the case $n=1$ in [14] and is analogous to the theory for relative equilibria in [15].

First introduce a linear change of coordinates,

$$[\mathbf{w}_1|\cdots|\mathbf{w}_{2n+2}] = [\mathbf{v}_1|\cdots|\mathbf{v}_{2n+2}]\mathbf{T},$$

where $\mathbf{T}$ is a $(2n+2)\times(2n+2)$ matrix. $\mathbf{T}$ is explicitly computable but its entries are not needed here. Let $s_1=\pm 1$ and let $s_2,\ldots,s_n$ be the signs of the nonzero eigenvalues of $\mathrm{D}I(\boldsymbol{\omega})$. The sign s_1 is determined from the top of the Jordan chain: it is related to the sign of $\langle\langle \mathbf{J}\mathbf{v}_{2n+2},\mathbf{v}_1\rangle\rangle$. The signs s_j are not important for the dynamics, but they are important for assuring that the transformation is symplectic.

Express the general solution of linear problem (13) in the form

$$\mathbf{u}(t) = \phi_1(t)\mathbf{w}_1 + \cdots + \phi_n(t)\mathbf{w}_n + u(t)\mathbf{w}_{n+1} - s_1\hat{I}_1(t)\mathbf{w}_{2n+2} + s_2\hat{I}_2(t)\mathbf{w}_{n+2} + \cdots + s_n\hat{I}_n(t)\mathbf{w}_{2n} + s_1 v(t)\mathbf{w}_{2n+1} + \cdots . \tag{15}$$

Then to leading order the normal form is

$$-\dot{\hat{I}}_j = 0 + \cdots \quad j = 1, \ldots, n,$$

$$-\dot{v} = \hat{I}_1 - \tfrac{1}{2}\kappa u^2 + \cdots ,$$

$$\dot{\phi}_1 = u + \cdots ,$$

$$\dot{\phi}_j = s_j \hat{I}_j + \cdots j = 1, \ldots, n,$$

$$\dot{u} = s_1 v + \cdots , \tag{16}$$

where $\hat{I}_1 = \mathcal{A}(\boldsymbol{\omega})$, where $\mathcal{A}$ from Eq. (7) is a measure of the distance from the action hypersurface. The coefficient κ in the normal form is determined by the second derivative of the action-frequency map

$$\kappa = \mathbf{C}\hat{\kappa},$$

where $\mathbf{C}$ is an explicitly computable positive constant that is related to the absolute value of $\langle\langle \mathbf{J}\mathbf{v}_{2n+2}, \mathbf{v}_1 \rangle\rangle$, and $\hat{\kappa}$ is defined in Eq. (6).

The Hamiltonian function associated with the leading-order normal form is

$$H = \sum_{j=2}^{n} s_j \hat{I}_j^2 + u\hat{I}_1 + \frac{1}{2}s_1 v^2 - \frac{1}{6}\kappa u^3 + \cdots .$$

In contrast to the leading-order normal form of Hanßmann (1), here the torus and normal direction are coupled at leading order through the term $u\hat{I}_1$, the other toral directions, $\hat{I}_2, \ldots, \hat{I}_n$, have a signature $\{s_2, \ldots, s_n\}$ determined by the signs of the nonzero eigenvalues of $\mathrm{D}I(\boldsymbol{\omega})$, and the coefficient of the nonlinear term κ is determined by the second derivative of the action-frequency map.

V. HOMOCLINIC BIFURCATION AND GEOMETRIC PHASE

There are two interesting features of the solution of the leading-order normal form: the bifurcating homoclinic orbit and the induced geometric phase along the torus. The homoclinic orbit to leading order is given by Eq. (15) with the coefficients determined from the nonlinear normal form. For example,

$$u(t) = \nu - 3\nu\,\mathrm{sech}^2(\gamma t),$$

with $\gamma = \frac{1}{2}\sqrt{s_1 a_0 \kappa \nu}$ and $\nu = \pm|\kappa|^{-1}\sqrt{2\kappa \hat{I}_1/a_0}$. Keep in mind that this is only the $u(t)$ coefficient of $\mathbf{w}_{n+1}(\boldsymbol{\theta})$ in Eq. (15) and so the flow on the torus has to be added in to get the full picture.

The geometric phase is determined by the first phase function $\phi_1(t)$ [coordinates have been chosen so that the degenerate direction is aligned with $\phi_1(t)$]. The geometric phase is determined by integrating the equation $\dot{\phi}_1(t) = u + \cdots$. With $u(t)$ above,

$$\phi_1(t) = \nu t - \frac{3\nu}{\gamma}\tanh(\gamma t) + \phi_1^0,$$

and so the geometric part of the phase shift is

$$\Delta\phi_1 = [\phi_1(t) - \nu t]_{-\infty}^{+\infty} = -6\nu/\gamma. \tag{17}$$

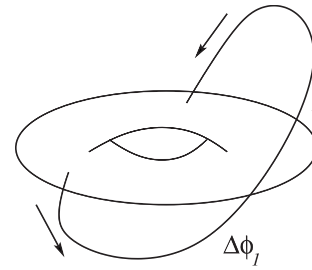

FIG. 2. Schematic of the phase shift of the orbit which is homoclinic to the manifold of invariant tori.

The geometric phase measures the gap—along the torus—between the unstable manifold leaving the torus and the stable manifold returning to the torus. The phase-space dimension is too high to effectively draw the picture, but a schematic is shown in Fig. 2. This geometric phase is analogous to a Hannay-Berry geometric phase [16], and generalizes the topological angle for homoclinics connecting to periodic orbits [14,17].

VI. OVERVIEW, APPLICATION, AND PERSISTENCE

The main observation of this paper is that the action-frequency map both determines where homoclinic bifurcation of invariant tori will occur via Eq. (4), and the nonlinear properties of the bifurcating homoclinic orbit, namely, the coefficient $\hat{\kappa}$. The singular hypersurfaces where Eq. (4) is satisfied are distinct from points where Eq. (2) is satisfied. Hence the theory of Hanßmann [12] and related work will apply in distinctly different regions of action space.

Given a family of invariant n tori, the properties of the homoclinic bifurcation are almost completely determined by studying just the action-frequency map. First look for hypersurfaces satisfying Eq. (4). Note that such hypersurfaces are not necessarily connected. Then compute the eigenvalues of $\mathrm{D}I(\boldsymbol{\omega})$ on the frequency hypersurface. The signs $s_2, \ldots, s_n$ are then the signs of the nonzero eigenvalues. Computing second derivative (6) then gives the coefficient of the nonlinear term in the normal form. The only property which is not given by the action-frequency map is the sign s_1. But this is obtained by a purely linear calculation as it involves an inner product between the first eigenvector, and the top eigenvector of the Jordan chain.

With the addition of nonintegrable terms, by increasing the dimension beyond $2n+2$ or by introducing symmetry-breaking terms, the theory still goes through formally, but small divisors will be present. This will affect both the smoothness of the action-frequency map and the persistence of the invariant tori. There has been much work on persistence of invariant tori in the presence of degeneracy, e.g., [18–22] and references therein. However, in all these cases they consider the degeneracy of the inverse map,

$$\det[\mathrm{D}\boldsymbol{\omega}(I)] = 0.$$

With the addition of secondary nondegeneracy conditions they are able to prove persistence of a subset of the invariant tori. Surprisingly, persistence of invariant tori in the case of inverse degeneracy (4) has not been considered. Hence the precise nature of the persistence of the invariant torus with attached homoclinic orbit found here, in the presence of perturbation, is an open question.

VII. EXAMPLE—TORAL DARK SOLITARY WAVES

Consider the coupled Ginzburg-Landau equations

$$\mathrm{i}A_t = A_{xx} + r_1 A + \alpha|A|^2 A + \beta|B|^2 A,$$

$$\mathrm{i}B_t = \mathrm{i}aB_x + r_2 B + \beta|A|^2 B + \gamma|B|^2 B, \tag{18}$$

which arise in pattern formation. In this equation $A(x,t)$ and $B(x,t)$ are complex valued, and the parameters r_1, r_2, a, α, β, and γ are all real. Assume $a \neq 0$ and $\alpha\gamma - \beta^2 \neq 0$. The steady equations can be characterized as a Hamiltonian system on $\mathbb{R}^6$. This Hamiltonian system has a two-parameter family of invariant two-tori (for fixed values of the external parameters) parameterized by the values of the actions or frequencies. This family is degenerate on a codimension-one hypersurface in frequency space, and near this curve of degeneracy a homoclinic orbit (bi-asymptotic to the two tori) is generated. In the spatial setting this homoclinic orbit represents a toral dark solitary wave, generalizing classical dark solitary waves which are bi-asymptotic to a periodic solution [23,24].

The representation of the steady equations as a Hamiltonian system on $\mathbb{R}^6$ proceeds as follows. Let $\mathbf{u} = (q_1, q_2, p_1, p_2, q_3, p_3)$ with

$$A = q_1 + \mathrm{i}q_2, \quad A_x = p_1 + \mathrm{i}p_2, \quad B = q_3 + \mathrm{i}p_3,$$

then the steady part of Eq. (18) is equivalent to

$$\mathbf{J}\mathbf{u}_x = \nabla H(\mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^6,$$

with

$$H(\mathbf{u}) = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}r_1(q_1^2 + q_2^2) + \tfrac{1}{2}r_2(q_3^2 + p_3^2) + \tfrac{1}{4}\alpha(q_1^2 + q_2^2)^2 + \tfrac{1}{2}\beta(q_1^2 + q_2^2)(q_3^2 + p_3^2) + \tfrac{1}{4}\gamma(q_3^2 + p_3^2)$$

and

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & -a & 0 \end{bmatrix}.$$

There is an exact toral solution

$$\mathbf{u}(x) = \mathbf{G}[\theta_1(x), \theta_2(x)]\mathbf{u}_0, \quad \frac{d\theta_1}{dx} = \omega_1, \quad \frac{d\theta_2}{dx} = \omega_2.$$

where

$$\mathbf{G}(\theta_1, \theta_2) := \mathbf{R}_{\theta_1} \oplus \mathbf{R}_{\theta_1} \oplus \mathbf{R}_{\theta_2}, \quad \mathbf{R}_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Percival's variational principle gives a relation between $\mathbf{u}_0$ and ω,

$$r_1 - \omega_1^2 + \alpha[(q_1^0)^2 + (q_2^0)^2] + \beta[(q_3^0)^2 + (p_3^0)^2] = 0,$$

$$r_2 - a\omega_2 + \beta[(q_1^0)^2 + (q_2^0)^2] + \gamma[(q_3^0)^2 + (p_3^0)^2] = 0, \tag{19}$$

where $\mathbf{u}_0 := (q_1^0, q_2^0, p_1^0, p_2^0, q_3^0, p_3^0)$.

Evaluation of actions (11) gives

$$I_1 = \tfrac{\omega_1}{\delta}(\gamma\omega_1^2 - a\beta\omega_2 - \gamma r_1 + \beta r_2),$$

$$I_2 = \tfrac{a}{2\delta}(-\beta\omega_1^2 + a\alpha\omega_2 + \beta r_1 - \alpha r_2), \tag{20}$$

where $\delta = \alpha\gamma - \beta^2$. The Jacobian is

$$\mathrm{D}I(\boldsymbol{\omega}) = \frac{1}{\delta}\begin{bmatrix} 3\gamma\omega_1^2 - a\beta\omega_2 - \gamma r_1 + \beta r_2 & -a\beta\omega_1 \\ -a\beta\omega_1 & \tfrac{1}{2}\alpha a^2 \end{bmatrix},$$

and so

$$\det[\mathrm{D}I(\boldsymbol{\omega})] = \tfrac{1}{2}\tfrac{a^2}{\delta^2}[(3\gamma\alpha - 2\beta^2)\omega_1^2 - a\alpha\beta\omega_2 + \alpha\beta r_2 - \alpha\gamma r_1].$$

Setting this determinant to zero generates a parabola in the frequency plane, for fixed values of the external parameters.

The requirements that

$$(q_1^0)^2 + (q_2^0)^2 > 0 \quad \text{and} \quad (q_3^0)^2 + (p_3^0)^2 > 0$$

define the region where invariant two-tori exist. Substitution into Eq. (19) gives the region of existence: tori exist for all frequencies satisfying

$$\delta(\gamma\omega_1^2 - a\beta\omega_2 - \gamma r_1 + \beta r_2) > 0,$$

$$\delta(-\beta\omega_1^2 + a\alpha\omega_2 + \beta r_1 - \alpha r_2) > 0. \tag{21}$$

At any point on the action hypersurface, the normal vector $\mathbf{n}$ is

$$\mathbf{n} = \mathsf{C}\begin{pmatrix} \alpha a \\ 2\beta\omega_1 \end{pmatrix}, \quad \mathsf{C} = (\alpha^2 a^2 + 4\beta^2\omega_1^2)^{-1/2}.$$

Let $\boldsymbol{\omega}$ be a fixed point on the frequency hypersurface and let $\mathbf{n}$ be the normal vector at that point. Then the coefficient $\hat{\kappa}$ defined in Eq. (6) is

$$\hat{\kappa} = \left.\frac{d^2}{ds^2}\right|_{s=0} \mathbf{n}\cdot I(\boldsymbol{\omega} + s\mathbf{n}) = 6\alpha^2 a^3\omega_1.$$

The sign s_2 is determined by

$$s_2 = \text{sign Trace}[\mathrm{D}I(\boldsymbol{\omega})],$$

with $\boldsymbol{\omega}$ in the frequency hypersurface. The sign s_1 is determined by computing the six generalized eigenfunctions. However, the sign s_1 does not affect the dynamics; it only

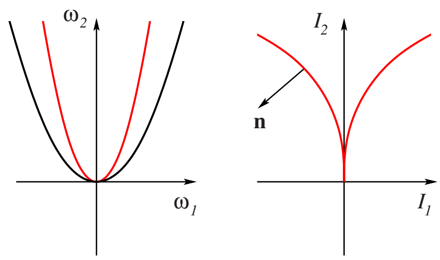

FIG. 3. (Color online) The frequency hypersurface (left figure) and action hypersurface (right figure) for example (20). The region of existence of invariant two tori is the interior of outer parabola in frequency space, and the inner parabola is the frequency hypersurface.

affects which side of the hypersurface the homoclinic bifurcation takes place.

To show that the set of existing tori and singular submanifold are nonempty, choose some parameters, e.g.,

$$a=+1, \quad \alpha=-1, \quad \beta=2, \quad \gamma=1, \quad r_1=-1, \quad r_2=+1.$$

Then solutions exist for all (ω_1,ω_2) such that

$$2\omega_2 > 3+\omega_1^2,$$

and the frequency hypersurface is the parabola

$$2\omega_2 = 3+11\omega_1^2.$$

This curve and the bounding curve for the existence set are shown in Fig. 3. In the left figure, the outer parabola is border of the existence region, and the inner parabola is the curve $\det[\mathrm{D}I(\boldsymbol{\omega})]=0$. The right figure shows the image of the singular curve in action space, and its normal vector. For these parameter values,

$$\mathbf{n} = \frac{1}{\sqrt{1+16\omega_1^2}}\begin{pmatrix}-1\\ 4\omega_1\end{pmatrix}, \quad s_2=+1, \quad \hat{\kappa}=6\omega_1.$$

Hence away from the point $\omega_1=0$ there is a homoclinic bifurcation in action space near the action hypersurface determined by the normal form expression in Sec. V. For the Ginzburg-Landau equation the homoclinic orbit is spatial and so it is a solitary wave. Since it is bi-asymptotic as $x\to\pm\infty$ to an invariant two-torus, it is called a toral dark solitary wave.

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